

Synchronization through time-shifted shared inputs

Ehsan Bolhasani,^{1,2} Yousef Azizi,¹ and Alireza Valizadeh^{1,2}

¹*Department of physics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran.*

²*School of Cognitive Sciences, Institute for Studies in Theoretical Physics and Mathematics, Niavaran, Tehran, Iran.*

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When oscillators in a population receive signal from common sources, their inputs will be statistically correlated which can result to synchronization of the oscillators. In real physical and biological systems finite speed of the signal transmission might result in the correlation of non-zero lags between inputs if the target oscillators are of different distance from the common sources. In this study we show that how presence of non-zero lags correlation between the stochastic inputs of two limit-cycle oscillators affects their relative phase. We show that the lag in the shared inputs can be formulated as the delay in the coupling terms and this in turn leads to nontrivial results when the neurons are coupled by direct connections and the neurons receive common stochastic inputs with a non-zero time shift.

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I. INTRODUCTION

Synchronized activity of the nonlinear oscillators has been observed in a wide variety of the physical and biological systems. Coordination of the activity of the populations of nonlinear oscillators can enhance their collective output which has been exploited by different biological and technological systems [1–3]. For example synchronized neurons produce different vital rhythms [4, 5] or they impact the target nervous systems with stronger signals [6].

Coordination of the activity of nonlinear oscillators might arise from the connections between the oscillators and/or through common external drives [7–9]. Presence of common source leads to statistical correlations between the inputs to the oscillators which increases interdependency of their activity and might lead to synchrony when the inputs are identical [10, 11]. In a realistic system, the transmission of the signals from common sources to the system are of finite speed and consequently the inclusion of the different transmission times are necessary if the common source is not of equal distance to the system constituent units. In this case, the input cross-correlation function for every pair of oscillators peaks in a finite time lag. In this study we aimed to inspect the effect of this input finite time shift on the output correlation function of two oscillators. The results for uncoupled oscillators expectedly show that the effect of such input time lag is a shift in the output cross-correlation function. For coupled oscillators, inputs time-lag non-trivially affects the output correlation function and besides the shift, the maximum correlation is itself dependent to the input time lag. By analytical calculations and numerical experiments we have shown that the degree of synchrony is dependent to the time-lag of shared inputs and maximum synchrony could be attained when the common inputs to the neurons are (differentially) shifted by a non-zero time lag.

We consider two bidirectionally coupled oscillators with state vector X_i , ($i = 1, 2$) which evolves according to

$$\begin{aligned}\dot{X}_1(t) &= F(X_1(t), I_1) \\ &\quad + \varepsilon g_{12} G_{12}(X_1(t), X_2(t - \tau_{12})) + \sqrt{D\varepsilon} \xi(t), \\ \dot{X}_2(t) &= F(X_2(t), I_2) \\ &\quad + \varepsilon g_{21} G_{21}(X_2(t), X_1(t - \tau_{21})) + \sqrt{D\varepsilon} \xi(t - \tau_s),\end{aligned}\tag{1}$$

where vector function $F(X_i, I_i)$ describes the inherent dynamic of the isolated oscillators with I_i being a current-like scalar parameter which determines the bifurcation of the oscillators from stationary to oscillatory regimes. We assume that with $I_i > I_{th}$ isolated oscillators have a stable limit cycle $X_{LC}(t) = X_{LC}(t + T)$ with period T a decreasing function of I_i once $I_i > I_{th}$. G_{ij} determines the interaction function whose strength is controlled by g_{ij} and acts through delay time τ_{ij} . The last terms in the equations describe common stochastic input to the two oscillators where $\xi(t)$ is Gaussian white noise with zero mean and unit variance and D determines the noise intensity. τ_s is the key parameter of the present study which determines the time shift between the two stochastic inputs, i.e., the second neuron receives the same input but before the time lag τ_s . Both the interaction term and stochastic inputs are scaled with the small parameter $\varepsilon \ll 1$. Furthermore the oscillators are assumed to be in general nonidentical due to the small difference in their bifurcation parameter $I_1 - I_2 = \varepsilon \Delta I$.

Applying the standard phase reduction method in the regime of weak perturbation [11, 12] leads to the following Itô stochastic differential equations, for the time evolution of the phase variable $\theta(X)$ in the vicinity of the

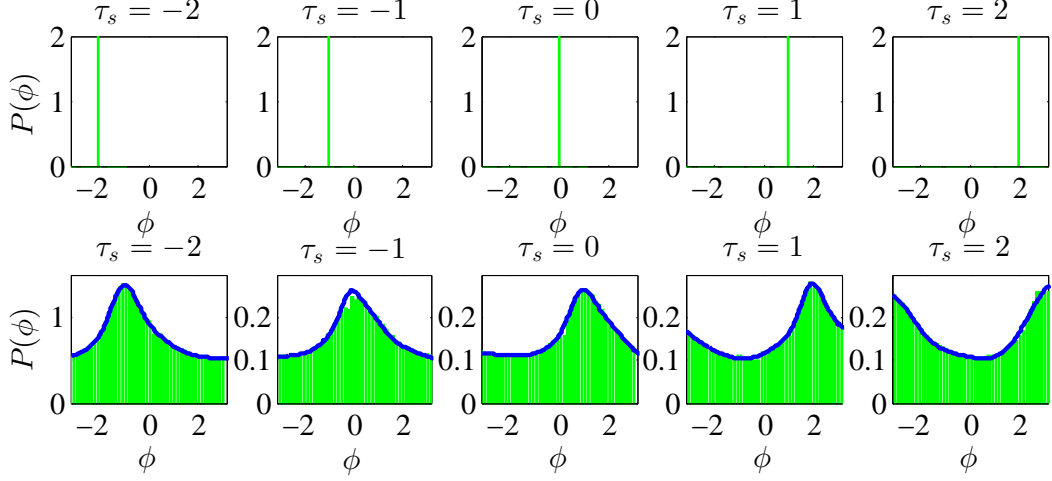


FIG. 1: (a) Phase difference distribution function of two identical uncoupled neurons, receiving a fully correlated Gaussian white noise with different time shifts shown above the plots. **(b)** Same results for non-identical oscillators with frequency mismatch $\Delta\omega = 0.5$ (Parameters: $D = 0.5$, and τ_s is $-2, -1, 0, 1, 2$ from left to right).

unperturbed limit cycle X_{LC}

$$\begin{aligned}\dot{\theta}_1(t) &= \omega_1 + \varepsilon g_{12} \mathbf{Z}(\theta_1) \cdot \mathbf{G}(\theta_1(t), \theta_2(t - \tau_{12})) \\ &\quad + \sqrt{D\varepsilon} \mathbf{Z}(\theta_1) \cdot \boldsymbol{\xi}(t) \\ \dot{\theta}_2(t) &= \omega_2 + \varepsilon g_{21} \mathbf{Z}(\theta_2) \cdot \mathbf{G}(\theta_2(t), \theta_1(t - \tau_{21})) \\ &\quad + \sqrt{D\varepsilon} \mathbf{Z}(\theta_2) \cdot \boldsymbol{\xi}(t - \tau_s)\end{aligned}\quad (2)$$

where $Z(\theta) = \nabla_X \theta|_{X_{LC}(\theta)}$ is the phase sensitivity [13]. Since we assumed small inhomogeneity in the bifurcation parameter $\Delta I \sim \mathcal{O}(\varepsilon)$, difference in natural frequencies will be also small $\omega_1 - \omega_2 = \varepsilon \Delta\omega$.

We take $\omega_1 = 1$ and $\omega_2 = 1 - \varepsilon \Delta\omega$ and define time-shifted phase difference as $\theta(t) = \theta_1(t) - \theta_2(t + \tau_s)$. From Eqs. 2 we obtain the evolution of $\theta(t)$ to the order of ε

$$\begin{aligned}\dot{\theta}(t) &= \varepsilon \Delta\omega + \varepsilon \left(H_{12}(\theta_1(t), \theta_2(t - \tau_{12})) \right. \\ &\quad \left. - H_{21}(\theta_2(t + \tau_s), \theta_1(t + \tau_s - \tau_{21})) \right) \\ &\quad + \sqrt{\varepsilon} \mathbf{f}(\theta_1(t), \theta_2(t + \tau_s)) \cdot \boldsymbol{\xi}(t)\end{aligned}\quad (3)$$

here $\mathbf{f}(\theta_1(t), \theta_2(t + \tau_s)) = \sqrt{D} \left[\mathbf{Z}(\theta_1(t)) - \mathbf{Z}(\theta_2(t + \tau_s)) \right]$ and $H_{ij}(\theta_i(t), \theta_j(t + \tau_s)) = g_{ij} \mathbf{Z}(\theta_i(t)) \cdot \mathbf{G}(\theta_i(t), \theta_j(t + \tau_s))$. We assume $\theta_i(t) = t + \varphi_i(t)$ where the first term t captures the intrinsic dynamics of isolated oscillators and the second term is slow varying deviation from natural oscillations. Averaging the equation over one period [14], we have

$$\begin{aligned}\frac{d\varphi(t)}{dt} &= \varepsilon \Delta\omega + \varepsilon \left(\bar{H}_{12}(\varphi - \tau_{12}) - \bar{H}_{21}(-\varphi + \tau_{21}) \right) \\ &\quad + \sqrt{\varepsilon} \bar{\mathbf{f}}(\varphi - \tau_s) \boldsymbol{\eta}(t)\end{aligned}\quad (4)$$

where $\varphi = \varphi_1 - \varphi_2$ and the averaged functions \bar{H}_{ij} and $\bar{\mathbf{f}}$ are defined as $\bar{H}_{ij}(\theta_i(t), \theta_j(t + \tau_s)) = \frac{1}{2\pi} \int_0^{2\pi} H_{ij}(\theta_i(t), \theta_j(t + \tau_s)) d\varphi$ and $\bar{\mathbf{f}}(\varphi - \tau_s) = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} dt \left[\mathbf{f}(\theta_1(t), \theta_2(t + \tau_s)) \right]^2}$. Note that in the above equation the lag in the common inputs τ_s acts like delay in the connections with a change of variable $\varphi \rightarrow \varphi + \tau_s$. We finally derive the Fokker-Planck equation for the distribution of the phase differences of the two oscillators, described by Eq. 4

$$\begin{aligned}\frac{\partial \rho}{\partial t}(\varphi, t) &= -\varepsilon \frac{\partial}{\partial \varphi} \left[\Gamma(\varphi) \rho(\varphi, t) \right] \\ &\quad + \varepsilon \frac{\partial^2}{\partial \varphi^2} \left(\bar{\mathbf{f}}(\varphi - \tau_s) \rho(\varphi, t) \right)\end{aligned}\quad (5)$$

where $\Gamma(\varphi) = \Delta\omega + \left(\bar{H}_{12}(\varphi - \tau_{12}) - \bar{H}_{21}(-\varphi + \tau_{21}) \right)$ and $\rho(\varphi, t)$ depicts the distribution of φ . The stationary distribution of the phase differences can be calculated by letting $\partial \rho / \partial t = 0$

$$\rho_0(\varphi) = \frac{e^{M(\varphi)}}{N \bar{\mathbf{f}}(\varphi - \tau_s)} \left[\frac{e^{-M(2\pi)} - 1}{\int_0^{2\pi} e^{-M(x)} dx} \int_0^\varphi e^{-M(x)} dx + 1 \right] \quad (6)$$

where $M(\varphi) = \int_0^\varphi \frac{\Gamma(x)}{\bar{\mathbf{f}}(x - \tau_s)} dx$ and N is a normalization factor. In the following the analytical results are obtained by calculation of the stationary distribution function from the above equation (see Supplementary material for the details of derivations).

To check the validity of the Eq. 4 and the corresponding solution of Fokker-Planck equation Eq. 6, we take $Z(\theta) = 1 - \cos(\theta)$ which is the canonical form of the phase sensitivity for the type-I oscillators near SNIC bifurcation [15]. Furthermore, we assume the oscillators are pulse-coupled, i.e., $G_{ij} = \Sigma_n \delta(t - t_j^n - \tau_{ij})$ in which δ is

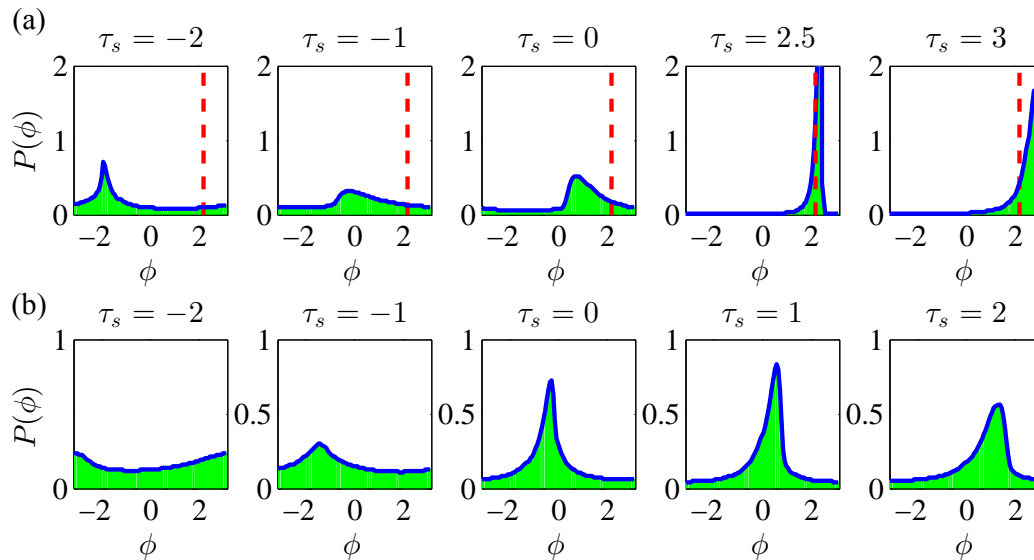


FIG. 2: (a) Phase difference distribution function of two coupled oscillators in the locked mode, receiving a fully correlated Gaussian white noise with different time shifts shown above each plot. (b) Same results for oscillators in the running mode. The results shown by green bar plots are calculated by numeric integration of Eq. 3 and the blue lines show the analytical result Eq. 6. The vertical dashed lines in (a) show the fixed point of Eq. 3 (Parameters: $D = 0.5$, $\Delta\omega = 0.2$ (for locked case), $\Delta\omega = -0.1$ (for running case)).

Dirac's delta function and t_j^n is the instant of $\theta_j = 2\pi n$. Pulse coupling approximation for interaction between oscillators is valid in the systems where the interaction term activates over a time which is small compared to period of the oscillation [16–19]. To assess the degree of synchrony regardless of the value of the phase lag we use the synchronization index $\gamma^2 = \langle \cos \varphi \rangle^2 + \langle \sin \varphi \rangle^2$, where the brackets denote the averaging over time [20].

First, we considered two uncoupled oscillators receiving a common noise with a time shift τ_s . For the identical oscillators $\Delta\omega = 0$, it has been shown that the oscillators synchronize when receiving common noises [11]. Our results show that time shift in the inputs results in the same time shift in the synchronized output without changing the synchrony index (Fig. 1a and Fig. 3). When the oscillators are not identical $\Delta\omega \neq 0$, phase difference distribution function spreads and synchrony index decreases (Fig. 1b and Fig. 3). In the presence of heterogeneity (mismatch of the firing rates) the most probable phase lag ϕ^* is not zero when the inputs have zero time-shift [21]. Interestingly this effect of heterogeneity can be compensated by a non-zero time shift in the inputs, i.e., the most probable phase difference of the oscillators could be around zero despite the heterogeneity by a suitable choice of the time lag of inputs (see Fig. 1b with $\tau_s \simeq -1$ and Fig. 3b). This means that the maximum zero-lag correlation of the two non-identical oscillators is achieved when the input to the high-frequency oscillator is lagged. Note that in this case changing the lag in the inputs does not change the functional form of the distribution function

and so the synchrony index.

Unlike the case of uncoupled oscillators the effect of time lag in the inputs to the coupled oscillators is not restricted to the shift of the distribution of the phase lags. For a system of two coupled oscillators, two cases can be recognized. In the first case (locked mode) the deterministic version of Eq. 4 (with no stochastic input) has a stable fixed point, and in the second case there is no fixed point for Eq. 4 and the system is in the *running mode*. The effect of common inputs in these two cases are shown in Fig. 2a and b, respectively. In the locked mode the synchrony index is no longer independent of the time lag of the inputs and peaks when the time lag coincides with the phase lag of deterministic case (Fig. 3). Accordingly, while the location of the peak of the distribution function shifts with changing input time lag, its maximum value and the width of the distribution around the most probable phase lag also changes (Fig. 2a). Analytical results obtained by the solving the stationary Fokker-Planck equation (Eq. 6) confirm the results of the simulation.

In the running mode the results are qualitatively similar to the locked mode except for the overall decrease in the synchrony index and wider distribution of the phase differences (Fig. 2b and Fig. 3). Changing the time-lag of the inputs the distribution is shifted while its width and maximum are changed. Again, the best entrainment with the maximum synchrony index is attained in a certain value of time lag of the inputs ($\tau_s \simeq 1$, see Fig. 2b and Fig. 3). Analytical results from Eq. 6 are again val-

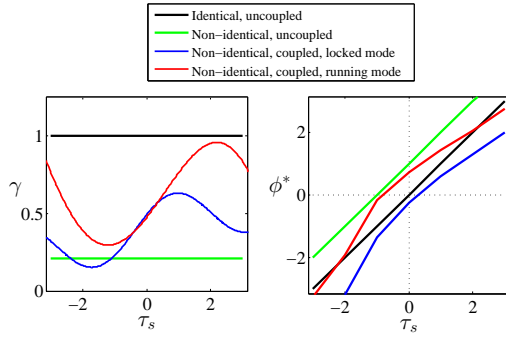


FIG. 3: (a) Synchronization index versus input time-shift for the case of two identical uncoupled phase oscillators (black), non-identical uncoupled phase oscillators (green), coupled phase oscillators in phase-locked mode (red), and non-identical coupled phase oscillators in the running mode (blue). (b) Most probable phase lag which shows the location of the peak of the phase lag distributions in Figs. 1 and 2 is plotted versus input time shift.

idated by the direct numeric solution of Eq. 3.

In summary, we have studied the synchronization of two limit cycle oscillators when the oscillators receive the common stochastic input with different time lags, i.e., the input to one of the oscillators is the same input to the other but shifted in time. We have shown that for uncoupled oscillators the effect of time lag is a trivial shift in the distribution of the phase difference of the two oscillators but when oscillators are coupled, input time lag changes the distribution of relative phases and so the degree of synchronization between the oscillators. Specifically with a *good* time lag around which the stochastic inputs accord with the most probable phase difference between the oscillators due to their deterministic dynamics, a resonance-like effect could be observed and the oscillators are more strongly phase locked.

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- [1] S. Steingrube, M. Timme, F. Wörgötter, and P. Manoonpong, *Nature physics* **6**, 224 (2010).
 - [2] K. Wiesenfeld, P. Colet, and S. H. Strogatz, *Physical review letters* **76**, 404 (1996).
 - [3] M. Rohden, A. Sorge, M. Timme, and D. Witthaut, *Physical review letters* **109**, 064101 (2012).
 - [4] D. M. Bramble and D. R. Carrier, *Science* **219**, 251 (1983).
 - [5] F. Yasuma and J.-i. Hayano, *Chest Journal* **125**, 683 (2004).
 - [6] Y. Ikegaya, G. Aaron, R. Cossart, D. Aronov, I. Lampl, D. Ferster, and R. Yuste, *Science* **304**, 559 (2004).
 - [7] A. Pikovsky, M. Rosenblum, J. Kurths, and R. C. Hilborn, *American Journal of Physics* **70**, 655 (2002).
 - [8] E. Bolhasani and A. Valizadeh, *Scientific reports* **5** (2015).
 - [9] E. Bolhasani, Y. Azizi, and A. Valizadeh, *Frontiers in Computational Neuroscience* **7**, 108 (2013).
 - [10] D. S. Goldobin and A. Pikovsky, *Physica A: Statistical Mechanics and its Applications* **351**, 126 (2005).
 - [11] J.-n. Teramae and D. Tanaka, *Phys. Rev. Lett.* **93**, 204103 (2004).
 - [12] Y. Kuramoto, *Chemical oscillations, waves, and turbulence* (Springer, Berlin, 1984).
 - [13] A. T. Winfree, *The geometry of biological time* (Springer, New York, 1980).
 - [14] G. B. Ermentrout and N. Kopell, *Journal of Mathematical Biology* **29**, 195 (1991).
 - [15] E. M. Izhikevich, *Dynamical systems in neuroscience* (MIT press, 2007).
 - [16] W. Gerstner, *Physical review letters* **76**, 1755 (1996).
 - [17] D. Hansel and G. Mato, *Physical Review Letters* **86**, 4175 (2001).
 - [18] C. S. Peskin, *Mathematical aspects of heart physiology* (Courant Institute of Mathematical Sciences, New York University, New York, 1975).
 - [19] S. H. Strogatz, *Nature* **410**, 268 (2001).
 - [20] P. Tass, M. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, and H.-J. Freund, *Physical review letters* **81**, 3291 (1998).
 - [21] S. D. Burton, G. B. Ermentrout, and N. N. Urban, *Journal of neurophysiology* **108**, 2115 (2012).